Functions

1. Definition.

A function is a mathematical relationship in which the values of a dependent variable are determined by the values of one or more independent variables.

Functions with a single independent variable are called Simple Univariate functions. There is a one to one correspondence.

Functions,with more than one independent variable, are called Multivariate functions.

The independent variable is often designated by X. The dependent variable is often designated by Y.

For example, Y is function of X which means Y depends on X or the value of Y is determined by the value of X. Mathematically one can write $Y = f(X)$.

2. Linear Equation

A statement of relationship between two quantities is called an equation. In an equation, if the largest power of the independent variable is one, then it is called as Linear Equation. Such equations when graphed give straight lines. For example $Y = 100-10X$.

For a straight line, there are two variables namely X and Y. X is called independent variable and Y is called dependent variable.

When value of 'X' increases by one unit, then the corresponding change in the 'Y' value is called as the slope of the line. Slope of the line is obtained by the formula,

m = slope (marginal change)

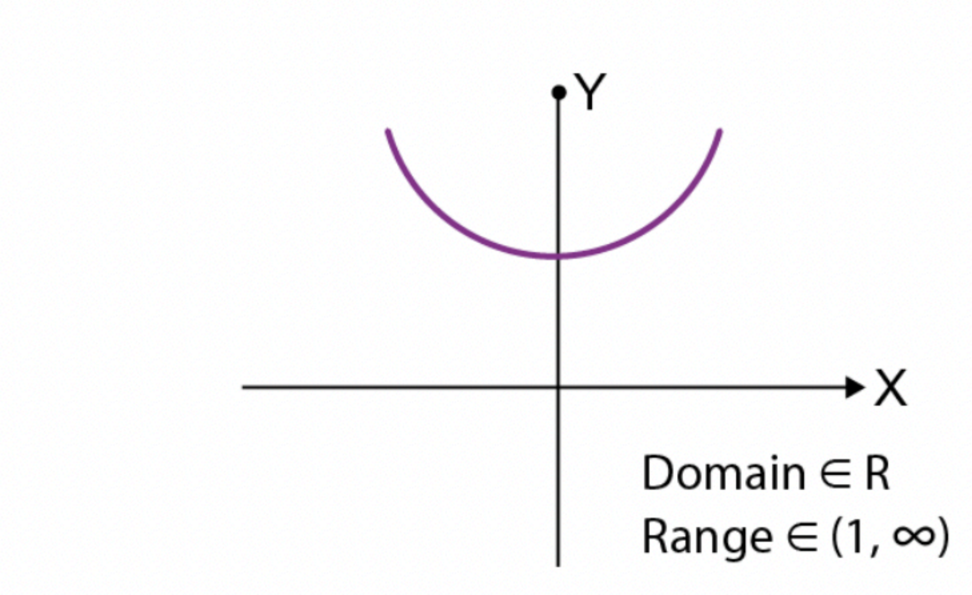
$$m = \frac{Y_2 - Y_1}{X_2 - X_1}, \frac{Change\ in\ Y}{Change\ in\ X}.$$

Where (X_1, Y_1) and (X_2, Y_2) are two arbitrary points

Slope or Gradient of the line represents the ratio of the changes in vertical and horizontal lines.

Quadratic Function

All functions in the form of $y = ax^2 + bx + c$ where a, b, c ∈ R, a ≠ 0 will be known as Quadratic function. The graph will be parabolic.



At $x = \frac{-b \pm \sqrt{D}}{2}$

, we will get its maximum on minimum value depends on the leading coefficient and that value will be -D/4a (where D = Discriminant)

In simpler terms,

A Quadratic polynomial function is a second degree polynomial, and it can be expressed as;

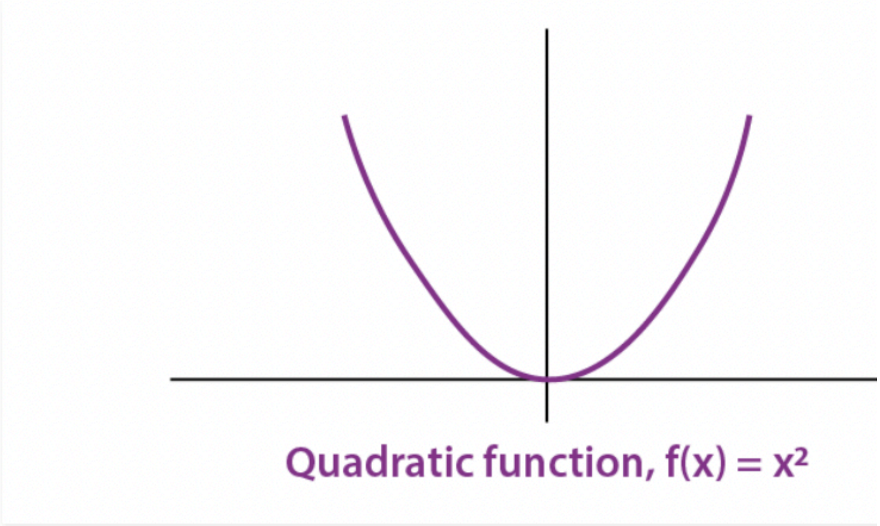
$F(x) = ax^2 + bx + c$, and a is not equal to zero.

Where a, b, c are constant, and x is a variable.

Example, $f(x) = 2x^2 + x - 1$ at x = 2

If x = 2, $f(2) = 2.2^2 + 2 - 1 = 9$

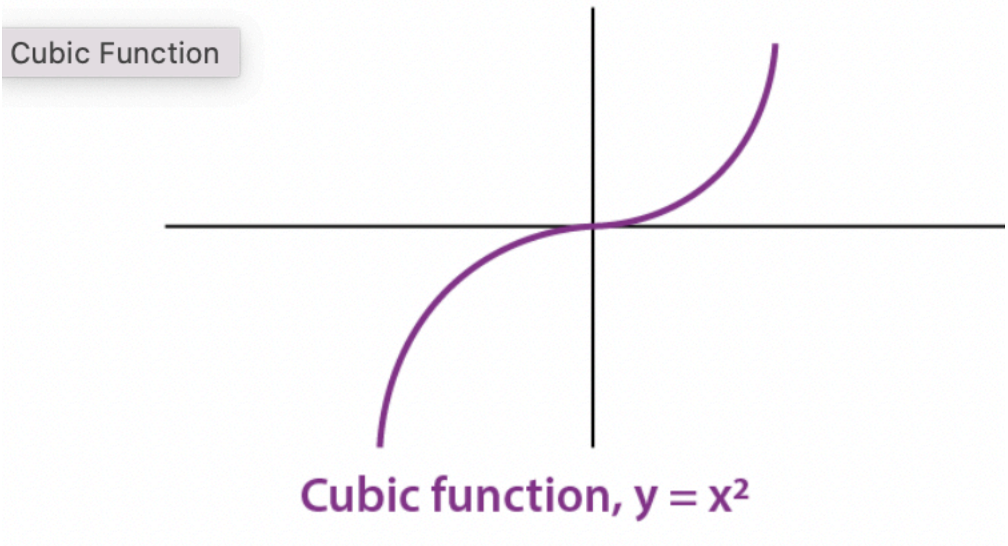
For Example: $y = x^2$



Cubic Function

A cubic polynomial function is a polynomial of degree three and can be expressed as;

$F(x) = ax^3 + bx^2 + cx + d$ and a is not equal to zero.



3. Application in Economics

By applying the above method, the demand and supply functions are obtained.

Demand Function: $Q_d = f(P_x)$ where Q_d stands for Quantity demand of a commodity and P_x is the price of that commodity.

Supply Function: $Q_s = f(P_x)$ where 'Q_s' stands for Quantity supplied of a commodity and Px is the price of that commodity.

In the example 12.1 the equation Y -5X + 12 has been obtained. It is a linear function.

Since slope is negative here, this function could be a demand function.

Demand Line

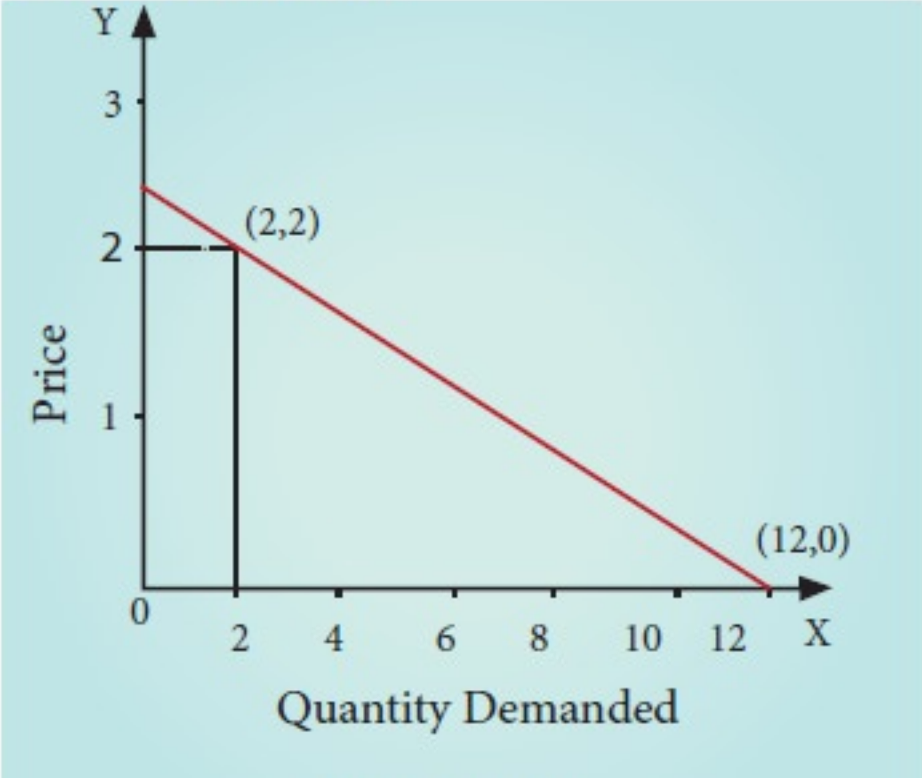


Diagram 12.2

Price-quantity relationship is negative in demand function.

$Q_d = 12-5X$ or $Qd = 12-5P$. If $P = 2$, $Qd = 2$.

When P assumes 0, only 12 alone remains in the equation. This is called Intercept or Constant, if $P = 0$ and $Qd = 12$.

In Marshallian analysis, money terms measured in Y-axis and physical units are measured in X-axis. Accordingly, price is measured in Y-axis and quantity demanded is measured in X-axis

Example: 12.2

Find the supply function of a commodity such that the quantity supplied is zero, when the price is **Rs.**5 or below and the supply (quantity) increases continuously at the constant rate of 10 units for each one rupee rise when the price is above **Rs.**5.

Solution:

To construct the linear supply function atleast two points are needed. First data point of supply function is obtained from the statement that the quantity supplied is zero, when the price is **Rs.**5, that is (0, 5).

The second and third data points of supply function can be obtained from the statement that supply increases 10 units for each one rupee rise in price, that is (10, 6) & (20, 7).

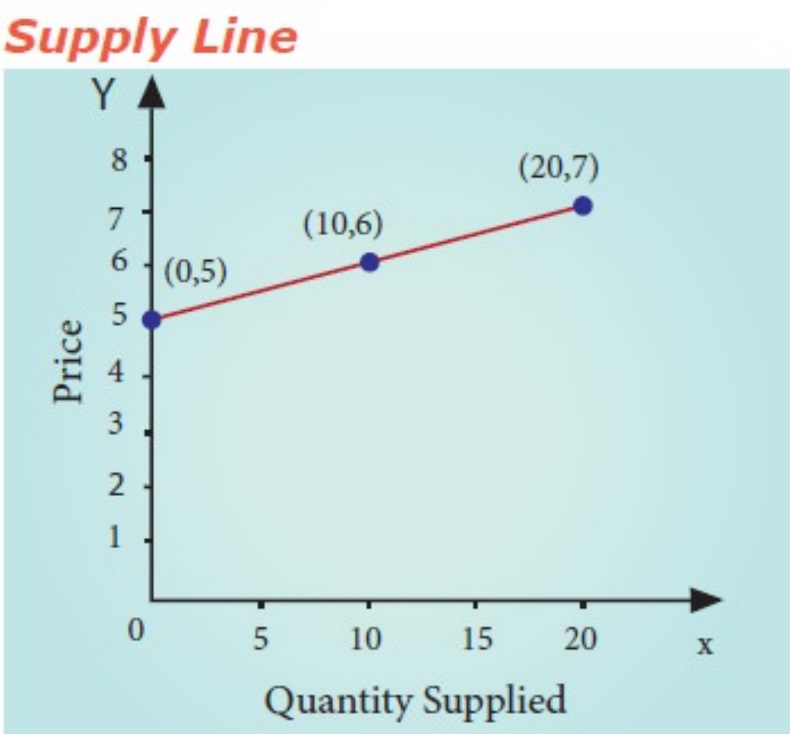


Diagram 12.3

When p = 5, supply is zero. When p = 6, supply is 10 and so on. When p is less than 5, say 4, supply is -10, which is possible in mathematics. But it is meaningless in Economics. Normally supply curve originates from zero, noting that when price is zero, supply is also zero.

4. Equilibrium

The point of intersection of demand line and supply line is known as equilibrium. The point of equilibrium is obtained by using the method of solving a set of equations. One can obtain the values of two unknowns with two equations. At equilibrium point,

Demand = Supply

(These are hypothetical examples)

$100-10p = 50 + 10\,p$

$100-50 = 20p$

$50 = 20p$

$p = 2.5$

When $p = 2.5$, Demand $= 100-10\,(2.5) = 75$

When $P = 2.5$, Supply $= 50 +10\,(2.5) = 75$

Differentiation Calculus:

Differential Calculus: The Concept of a Derivative:

The derivative dY /dX or more precisely the first derivative of a function is defined as limit of the ratio ΔY /ΔX as ΔX approaches zero. Thus

dY/dX = limit ΔX→0 ΔY/ΔX

The derivative of a function shows the change in value of the dependent variable one unit change in the independent variable (ΔX). Note that derivative of a function [Y=f (X)] is also written as d (fX) / dX. Or f' (X).

This may be explained with the help of following diagram

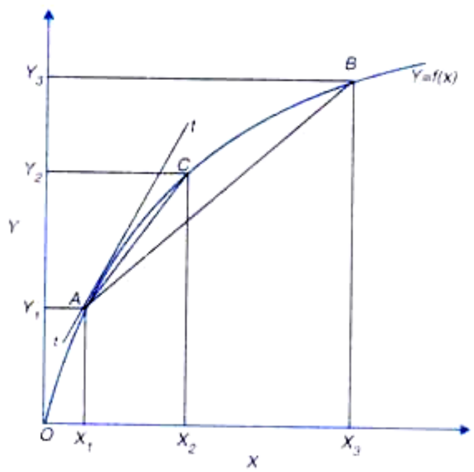


Fig 5.6. *Finding Derivative of a Function (Y = f(X) at a point*

Thus, derivative dY/dX is slope of a function whether it is linear or non-linear and represents a change in the dependent variable due to a small change in the independent variable. The concept of a derivative is extensively used in economics and managerial decision making, especially in solving the problems of optimisation such as those of profit maximisation, cost minimisation, output and revenue maximisation.

Rules of Differentiation:

Process of finding the derivative of a function is called differentiation. As stated above, derivative of a function represents the change in the dependent variable due to a infinitesimally small change in the independent variable and is written as dY / dX for a function Y = f (X). A series of rules have been derived for differentiating various types of functions. We describe below these rules of differentiation.

Derivative of a Constant Function:

A constant function is expressed as

$Y=f(X) = a$

Where 'a' is constant. The constant 'a' implies that Y does not vary as X varies, that is. Y is independent of X. Therefore, the derivative of a constant function is equal to zero. Thus, in this constant function

$dy/dx = 0$

For example, let the constant function be Y = 2.5

This is graphed in Figure 5.7(a). It will be seen that a constant function is a horizontal straight line (having a zero slope) which shows that irrespective of the value of the variable X, the value of Y does not change at all. Therefore, derivative dY/dX = 0.

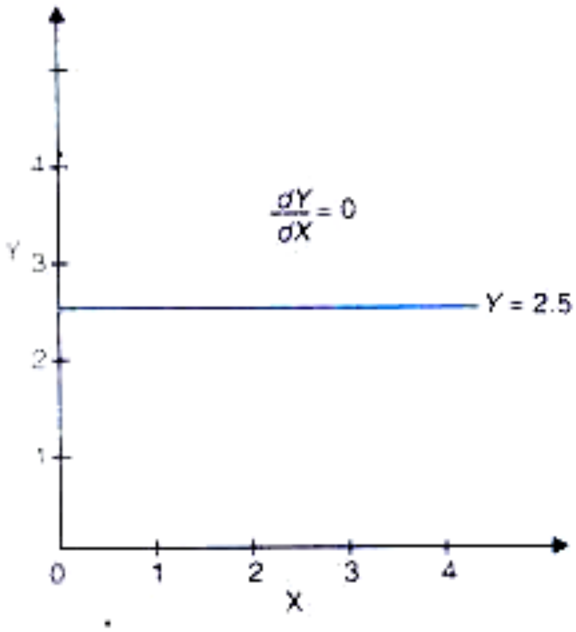


Fig. 5.7. (a) *Graph of a Constant Function*

Derivative of a Power Function:

A power function takes the following form:

$Y = aX^b$

Where a and b are constants. Here a is the coefficient of the X term and the variable X is raised to the power

b. The derivative of this power function is equal to the power b multiplied by the coefficient a times the variable X raised to the power b – 1. Thus rule for the derivative of power function ($Y = a X^b$) i

$$dY / dX = b. a. X^{b-1}$$

Let us take some examples of determining the derivative of a power function.

First, take the following power function:

Y = 1.5 X

In this function 1.5 is the coefficient of variable X, that is, a and the power b of X is 1 (implicit). Using the above rule for the derivative of a power function we have

$$dY / dX = 1 X 1.5 X^{1-1} = 1 X 1.5 X 0 = 1.5 X^0 = 1.5$$

This is graphically shown in Figure 5.7(b). It will be seen from this figure that slope of the linear function (Y = 1.5 X) is constant and is equal to 1.5 over any range of the values of the variables X.

Quadratic Power Function:

Let us take the following example of a power function which is of quadratic type:

$Y=X^2$

Its derivative, $dy/ dx = 2X^{2-1} = 2X^1 = 2X$

To illustrate it we have calculated the values of Y, associated with different values of X such as 1, 2, 2.5 and -1, -2, -2.5 and have been shown in Table 5.3.

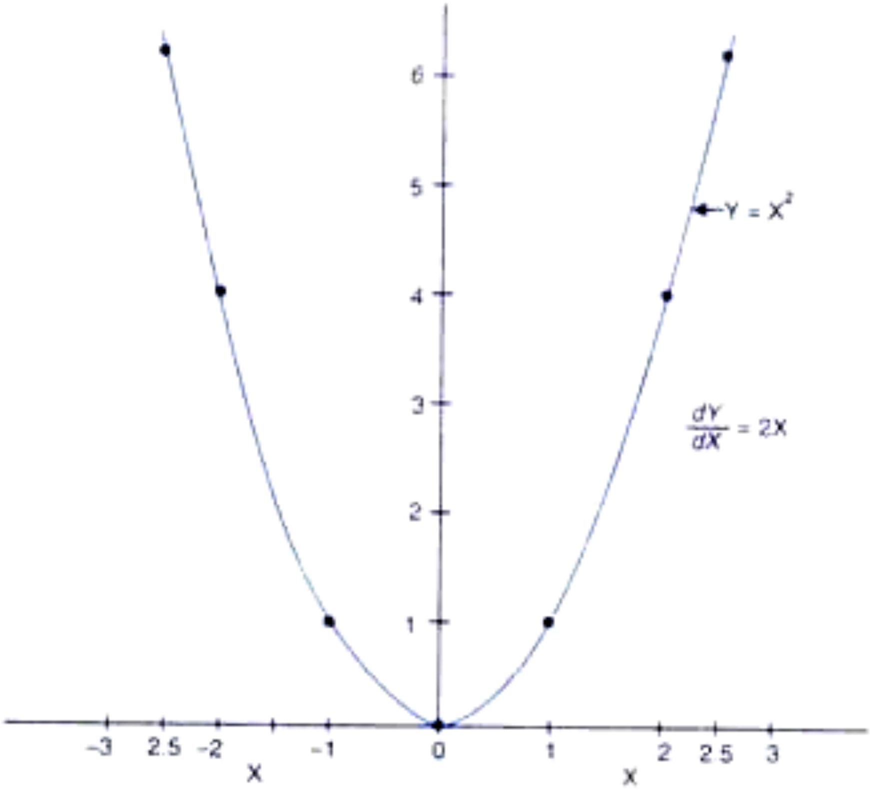


Fig. 5.8. Graph of a Quadratic function

We have plotted the values of X and corresponding values of Y to get a U-shaped parabolic curve in Figure 5.8. It will be seen that derivative dY / dX or, in other words, slope of this quadratic function is changing at different values of X.

Some other examples of power function and their derivatives are:

For power function,

$$Y = 3X^2$$

$$\frac{dy}{dx} = 2 \times 3.X^{2-1} = 6X$$

For power function:

$$Y = X^5$$

$$\frac{dY}{dX} = 5 \times 1.X^{5-1} = 5X^4$$

For power function:

$$Y = X$$

$$\frac{dY}{dX} = 1 \times 1\,X^{1-1} = 1 \times 1.X^0 = 1$$

It should be noted that any variable raised to the zero power (as in our example X^0) is equal to 1

For power function,

$Y = 3X^{-2}$

$dY / dX = -2 \times 3.X^{-2-1} = -6X^{-3}$

Derivative of a Sum or Difference of Two Functions:

Suppose, $Y = f(X) + g(X)$

Where $f(X)$ and $g(X)$ are the two unspecified functions and Y is the sum of the two functions. Then the derivative of their sum is

$$\frac{dY}{dX} = \frac{df(X)}{dX} + \frac{dg(X)}{dX}$$

Likewise, the derivative of the difference of the two or more different functions is the difference of their separate derivatives. Thus, if

$$Y = f(X) - g(X)$$

then $$\frac{dY}{dX} = \frac{df(X)}{dX} - \frac{dg(X)}{dX}$$

To illustrate, we take some examples.

If, $$Y = 4X^2 + 5X$$

then $$\frac{dY}{dX} = 2 \times 4X^{2-1} + 1 \times 5X^{1-1} = 8X + 5$$

If $$Y = 5X^2 - 2X^5$$

$$\frac{dY}{dX} = 10X - 10X^4$$

Now, consider the following profit function where each of the three terms represents a function

$$\pi = -40 + 140Q - 10Q^2$$

Where π stands for profit and Q for level of output. Then, derivative of profit (π) with respect to output (Q) is

$$\frac{d\pi}{dQ} = 0 + 140 - 20\,Q$$

$$= 140 - 20\,Q$$

Note that derivative of a constant (–40) is zero, derivative of 140 Q is 140 and derivative $10\,Q^2 = 20\,Q$.

The derivative of a sum of the two functions is equal to the sum of the derivatives obtained separately of the two functions.

Derivative of a Product of the Two Functions:

Suppose Vis the product of the two separate functions f (X) and g(X).

Y=f (X). g (X)

The derivative of the product of these two functions is equal to the first function multiplied by the derivative of the second function plus the second function multiplied by the derivative of the first function. Thus,

$$\frac{dY}{dX} = f(X).\frac{dg(X)}{dX} + g(X).\frac{df(X)}{dX}$$

For example, take the following function

$$Y = 5X^2 (2X + 3)$$

$f(X) = 5X^2$ and $g(X) = (2X + 3)$ are the two functions. Then, derivative of the product of these two functions,

$$\frac{dY}{dX} = 5X^2.\frac{d(2X+3)}{dX} + (2X+3).\frac{d(5X^2)}{dX}$$

$$\frac{dY}{dX} = 5X^2.2 + (2X + 3).10X$$

$$= 10X^2 + 20X^2 + 30X$$

$$= 30X^2 + 30X$$

$$= 30(X^2 + X)$$

Take another example of the product rule. Let

$$Y = (X^3 + X^2 + 5)(2X^2 + 3)$$

then, $$\frac{dY}{dX} = (X^3 + X^2 + 5).\frac{d(2X^2+3)}{dX} + (2X^2 + 3).\frac{d(X^3 + X^2 + 5)}{dX}$$

$$= (X^3 + X^2 + 5).4X + (2X^2 + 3).(3X^2 + 2X)$$

$$= (4X^4 + 4X^3 + 20X) + 6X^4 + 9X^2 + 4X^3 + 6X$$

$$= 10X^4 + 8X^3 + 9X^2 + 26X$$

Derivative of the Quotient of the Two Functions:

Suppose the variable 7is equal to the quotient of the two functions f (X) and g(X). That is,

Y =f (X) / g(X)

$$= \frac{(X-1)\times 5-(5X+2)\times 1}{(X-1)^2}$$

$$\frac{dY}{dX} = \frac{(5X-5)-(5X+2)}{(X-1)^2}$$

$$= \frac{dY}{dX} = \frac{5X-5-5X-2}{(X-1)^2}$$

$$= \frac{dY}{dX} = \frac{-7}{(X-1)^2}$$

Take another example. Let

$$Y = \frac{5-2X}{2X^2}$$

then,

$$\frac{dy}{dx} = \frac{2x^2 . \frac{d(5-2X)}{dX} - (5-2X) . \frac{d(2X^2)}{dX}}{(2X^2)^2}$$

$$= \frac{2X^2 . (-2) - (5-2X) . 4X}{4X^4}$$

$$= \frac{-4X^2 - 20X + 8X^2}{4X^4}$$

$$= \frac{4X^2 - 20X}{4X^4} = \frac{4X(X-5)}{4X(X^3)}$$

$$= \frac{X-5}{X^3}$$

Derivative of Function of a Function (Chain Rule):

When a variable Y is function of a variable U which in turn is related to another variable X, and if we wish to obtain a derivative of Y with respect to X, then we use chain rule for this purpose. Suppose variable Vis a function of the variable U, that is, Y = f (U) and variable U is a function of variable X, that is, U = g (X). Then, to obtain the derivative of Y with respect to X, that is dY / dX, we first find the derivative of the two functions, Y = f(U) and U = g(X) separately and then multiply them together. Thus,

For function, $Y = f(U)$

$$\frac{dY}{dU} = \frac{df(U)}{dU}$$

For function, $U = g(X)$

$$\frac{dU}{dX} = \frac{dg(X)}{dX}$$

Thus, according to the chain rule if Y= f(U) and U = g(X), then derivative of Y with respect X, can be obtained by multiplying together the derivative of Y with respect to U and the derivative of U with respect to X

Let us take some examples to illustrate this chain rule

Suppose $Y = U^3 + 15$ and $a = 3X^2$

Let us take some examples to illustrate this chain rule

Suppose $Y = U^\beta + 15$ and $U = 3X^2$

$$\frac{dY}{dX} = \frac{dY}{dU} \cdot \frac{dU}{dX}$$

then $$\frac{dY}{dX} = \frac{d(U^\beta + 15)}{dU} \cdot \frac{d(3X^2)}{dX} = 3U^2 \cdot 6X$$

Substituting $U = 3X^2$ in the above we have

$$\frac{dY}{dX} = 3.(3X^2)^2 \cdot 6X$$

$$= 3 \cdot 9X^4 \cdot 6X$$

$$= 27X^4 \cdot 6X$$

$$= 162X^5$$

Integration

Integration of Constant

Integration of constant function say ‘a’ will result in:

$\int a \, dx = ax + C$

Example:

$\int 4 \, dx = 4x + C$

Integration of Variable

If x is any variable then;

$\int x \, dx = x2/2 + C$

Integration of Square

If the given function is a square term, then;

$\int x2 \, dx = x3/3$

Application of Integration in Economics and Commerce.

Integration helps us to find out the total from the marginal. It is possible to find out consumer's surplus and producer's surplus from the demand and supply function. Cost and revenue functions are calculated through indefinite integral.

We learnt already that the marginal function is obtained by differentiating the total cost function. Now we shall obtain the total cost function when marginal cost function is given, by integration.

1. Cost functions from marginal cost functions

2. Revenue functions from Marginal revenue functions

3. The demand functions from elasticity of demand

4. Consumer's surplus

5. Producer surplus

Cost functions from marginal cost functions

If C is the cost of producing an output x, then marginal cost function MC = dc/dx

Using integration, as the reverse process of differentiation, we obtain,

Cost function $C = \int (MC)\, dx + k$

Where k is the constant of integration which is to be evaluated,

Average cost function AC = C/X, $x \neq 0$

Example 3.9

The marginal cost function of manufacturing x shoes is $6 + 10x - 6x^2$. The cost producing a pair of shoes is ₹12. Find the total and average cost function.

Solution:

Given,

Marginal cost $MC = 6 + 10x - 6x^2$

$C = \int MC\, dx + k$

$= \int (6 + 10x - 6x^2)\, dx + k$

$= 6x + 5x^2 - 2x^3 + k$ *(1)*

When $x = 2$, $C = 12$ (given)

$12 = 12 + 20 - 16 + k$

$k = -4$

$C = 6x + 5x^2 - 2x^3 - 4$

$$\text{Average cost} = \frac{C}{x} = \frac{6x + 5x^2 - 2x^3 - 4}{x}$$

$$= 6 + 5x - 2x^2 - \frac{4}{x}$$

Example 3.10

A company has determined that the marginal cost function for a product of a particular commodity is given by $MC = 125 + 10x - x2/9$ where C rupees is the cost of producing x units of the commodity. If the fixed cost is ₹250 what is the cost of producing 15 units.

Solution:

$$MC = 125 + 10x - \frac{x^2}{9}$$

$$C = \int MC\, dx + k$$

$$= \int \left(125 + 10x - \frac{x^2}{9}\right) dx + k$$

$$= 125x + 5x^2 - \frac{x^3}{27} + k$$

Fixed cost $k = 250$

$$C = 125x + 5x^2 - \frac{x^3}{27} + 250$$

When $x = 15$

$$C = 125\,(15) + 5\,(15)^2 - \frac{(15)^3}{27} + 250$$

$$= 1875 + 1125 - 125 + 250$$

$$C = ₹3{,}125$$

Example 3.11

The marginal cost function $MC = 2 + 5ex$ (i) Find C if $C(0)=100$ (ii) Find AC.

Solution:

Given $MC = 2 + 5e^x$

$$C = \int MC\,dx + k$$

$$= \int (2 + 5e^x)\,dx + k$$

$$= 2x + 5e^x + k$$

$$x = 0 \Rightarrow C = 100,$$

$$100 = 2(0) + 5(e^0) + k$$

$$\boxed{k = 95}$$

$$C = 2x + 5e^x + 95.$$

$$\text{Average cost} = \frac{C}{x} = \frac{2x + 5e^x + 95}{x}$$

$$AC = 2 + \frac{5e^x}{x} + \frac{95}{x}.$$

Revenue functions from Marginal revenue functions

If R is the total revenue function when the output is x, then marginal revenue $MR = dR/dx$ Integrating with respect to ' x ' we get

Revenue Function, $R = \int (MR)\, dx + k$.

Where 'k' is the constant of integration which can be evaluated under given conditions, when $x = 0$, the total revenue $R = 0$,

Demand Function, P=R/x, $x \neq 0$

Example 3.16

For the marginal revenue function $MR = 35 + 7x - 3x^2$, find the revenue function and demand function.

Solution:

Given $MR = 35 + 7x - 3x^2$

$$R = \int (MR)\, dx + k$$

$$= \int \left(35 + 7x - 3x^2\right) dx + k$$

$$R = 35x + \frac{7}{2}x^2 - x^3 + k$$

Since R = 0 when $x = 0$, $k = 0$

$$R = 35x + \frac{7}{2}x^2 - x^3$$

Demand function $P = \frac{R}{x}$

$$P = 35 + \frac{7}{2}x - x^2.$$

To find the Maximum Profit if Marginal Revenue and Marginal cost function are given:

If 'P' denotes the profit function, then $\frac{dP}{dx} = \frac{d}{dx}(R - C) = \frac{dR}{dx} - \frac{dC}{dx} = MR - MC$ Integrating both sides with respect to x gives , $P = \int(MR - MC)\,dx + k$

Where k is the constant of integration. However if we are given additional information, such as fixed cost or loss at zero level of output, we can determine the constant k. Once P is known, it can be maximum by using the concept of maxima and minima.

Example 3.18

The marginal cost $C'(x)$ and marginal revenue $R'(x)$ are given by $C'(x) = 50 + x/50$ *and* $R'(x) = 60$. The fixed cost is ₹200. Determine the maximum profit.

Solution:

$$\text{Given} \quad C(x) = \int C'(x)\,dx + k_1$$

$$= \int\left(50 + \frac{x}{50}\right)dx + k_1$$

$$C(x) = 50x + \frac{x^2}{100} + k_1$$

When quantity produced is zero, then the fixed cost is 200.

i.e. When $x = 0, c = 200$

$k1 = 200$

Cost function is $C(x) = 50x + \frac{x^2}{100} + 200$ (1)

The Revenue $R'(x) = 60$

$$R(x) = \int R'(x)\,dx + k_2$$

$$= \int 60\,dx + k_2$$

$$= 60x + k_2$$

When no product is sold, revenue = 0

i.e. When $x = 0, R = 0.$

Revenue $R(x) = 60x$ (2)

Profit P = Total Revenue – Total cost

$$= 60x - 50x - \frac{x^2}{100} - 200$$

$$= 10x - \frac{x^2}{100} - 200$$

$$\frac{dp}{dx} = 10 - \frac{x}{50}$$

To get profit maximum, $\frac{dp}{dx} = 0 \Rightarrow \boxed{x = 500.}$

$$\frac{d^2P}{dx^2} = \frac{-1}{50} < 0$$

$\therefore$ Profit is maximum when $x = 500$ and

Maximum Profit is $P = 10(500) - \frac{(500)^2}{100} - 200$

$=5000 - 2500 - 200$

$=2300$

Profit = ₹ 2,300.

Example 3.19

The marginal cost and marginal revenue with respect to commodity of a firm are given by $C'(x) = 8 + 6x$ and $R'(x) = 24$. Find the total Profit given that the total cost at zero output is zero.

Solution:

Given $\quad MC = 8 + 6x$

$$C(x) = \int (8+6x)\,dx + k_1$$

$$= 8x + 3x^2 + k_1 \qquad (1)$$

But given when $x = 0, C = 0 \Rightarrow k_1 = 0$

$$\therefore \quad C(x) = 8x + 3x^2 \qquad (2)$$

Given that $\quad MR = 24$

$$R(x) = \int MR\,dx + k_2$$

$$= \int 24\,dx + k_2$$

$$= 24x + k_2$$

Revenue = 0, when $x = 0 \Rightarrow k_2 = 0$

$$R(x) = 24x \qquad (3)$$

Total Profit functions $\quad P(x) = R(x) - C(x)$

$$P(x) = 24x - 8x - 3x^2$$

$$= 16x - 3x^2$$

Example 3.20

The marginal revenue function (in thousand of rupees) of a commodity is $10 + e^{-0.05x}$ Where x is the number of units sold. Find the total revenue from the sale of 100 units ($e^{-5} = 0.0067$)

Solution:

Given, Marginal revenue $R'(x) = 10 + e^{-0.05x}$

Total revenue from sale of 100 units is

$$R = \int_0^{100} \left(10 + e^{-0.05x}\right) dx$$

$$= \left[10x + \frac{e^{-0.05x}}{-0.05}\right]_0^{100}$$

$$= \left(1000 - \frac{e^{-5}}{0.05}\right) - \left(0 - \frac{100}{5}\right)$$

$$= 1000 + 20 - (20 \times 0.0067)$$

$$= 1019.87$$

Total revenue = 1019.87 × 1000

= ₹10,19,870

Inventory :

Given the inventory on hand $I(x)$ and the unit holding cost (C_1), the total inventory carrying cost is $C_1 \int_0^T I(x)dx$, where T is the time period under consideration.

Example 3.22

A company receives a shipment of 200 cars every 30 days. From experience it is known that the inventory on hand is related to the number of days. Since the last shipment, $I(x) = 200 - 0.2x$. Find the daily holding cost for maintaining inventory for 30 days if the daily holding cost is ₹3.5

Solution:

Here $I(x) = 200 - 0.2x$

$C_1 = ₹\ 3.5$

$T = 30$

$$\text{Total inventory carrying cost} = C_1 \int_0^r I(x)dx = 3.5\int_0^{30} (200 - 0.2x)dx$$

$$= 3.5\left(200x - \frac{0.2x^2}{2}\right)_0^{30} = 20{,}685$$

Index Numbers:

Meaning of Index Numbers:

It is a statistical device that measure the changes in the general price level.

Price index number indicates the average of changes in the prices of representative commodities at one time in comparison with that at some other time taken as the base period.

Steps in the Construction of Price Index Numbers:

1. Selection of Base Year:

The base year is defined as that year with reference to which the price changes in other years are compared and expressed as percentages. The base year should be a normal year.

2. Selection of Commodities:

Only representative commodities should be selected keeping in view the purpose and type of the index number.

3. Collection of Prices:

(a) Prices are to be collected from those places where a particular commodity is traded in large quantities.

(b) Published information regarding the prices should also be utilised,

(c) Information about prices should not be not biased.

(d) Selection of wholesale or retail prices depends upon the type of index number to be prepared.

(e) Prices collected from various places should be averaged.

4. Selection of Average:

Since the index numbers are, a specialised average, therefore suitable average to be chosen . Theoretically, geometric mean is the best for this purpose. But, in practice, arithmetic mean is used because it is easier to follow.

5. Selection of Weights:

Generally, all the commodities included in the construction' of index numbers are not of equal importance. Therefore, if the index numbers are to be representative, proper weights should be assigned to the commodities according to their relative importance.

For example, the prices of books will be given more weightage while preparing the cost-of-living index for teachers than while preparing the cost-of-living index for the workers. Weights should be unbiased and be rationally and not arbitrarily selected.

Construction of Price Index Numbers

1. Simple Aggregative Method:

In this method, the index number is equal to the sum of prices for the year for which index number is to be found divided by the sum of actual prices for the base year.

Formula:-

$$P_{01} = \frac{\Sigma P_1}{\Sigma P_0} \times 100$$

Where P_{01} Stands for the index number

ΣP_1 Stands for the sum of the prices for the year for which index number is to be found :

ΣP_0 Stands for the sum of prices for the base year.

Commodity	Prices in Base Year 1980 (in Rs.) P_o	Prices in current Year 1988 (in Rs.) P_1
A	10	20
B	15	25
C	40	60
D	25	40
Total	$\Sigma P_0 = 90$	$\Sigma P_1 = 145$

Index Number $(P_{01}) = \frac{\Sigma P_1}{\Sigma P_0} \times 100$; $P_{01} = \frac{145}{90} \times 100$; $P_{01} = 161.11$

2. Simple Average of Price Relatives Method:

In this method, the index number is equal to the sum of price relatives divided by the number of items and is calculated by using the following formula:

$$P_{01} = \frac{\Sigma R}{N}$$

Where ΣR stands for the sum of price relatives i. e. $R = \frac{P_1}{P_0} \times 100$ and

N stands for the number of items.

Example

Commodity P_0	Base Year Prices (in Rs.) P_1	Current year Prices (in Rs.)	Price Relatives $R = \frac{P_1}{P_0} \times 100$
A	10	20	$\frac{20}{10} \times 100 = 200.0$
B	15	25	$\frac{25}{15} \times 100 = 166.7$
C	40	60	$\frac{60}{40} \times 100 = 150.00$
D	25	40	$\frac{40}{25} \times 100 = 160.0$
N = 4			$\Sigma R = 676.7$

$$\text{Index Number } (p_{01}) = \frac{\Sigma R}{N}$$

$$P_{01} = \frac{676.7}{4} ; P_{01} = 169.2$$

3. Weighted Aggregative Method:

In this method, different weights are assigned to the items according to their relative importance. Weights used are the quantity weights. Many formulae have been developed to estimate index numbers on the basis of quantity weights.

Some of them are explained below:

(*i*) **Laspeyre's Formula.** In this formula, the quantities of base year are accepted as weights.

$$P_{01} = \frac{\Sigma P_1 q_0}{\Sigma P_0 q_0} \times 100$$

Where P_1 is the price in the current year ; P_0 is the price in the base year ; and q_0 is the quantity in the base year.

(*ii*) **Paasche's Formula.** In this formula, the quantities of the current year are accepted as weights.

$$P_{01} = \frac{\Sigma P_1 q_1}{\Sigma P_0 q_1} \times 100$$

Where q_1 is the quantity in the current year.

(*iii*) **Dorbish and Bowley's Formula.** Dorbish and Bowley's formula for estimating weighted index number is as follows :

$$P_{01} = \frac{\frac{\Sigma P_1 q_0}{\Sigma P_0 q_0} + \frac{\Sigma P_1 q_1}{\Sigma P_0 q_1}}{2} \times 100 \quad \text{or} \quad p_{01} = \frac{L + P}{2}$$

Where L is Laspeyre's index and P is paasche's Index.

(*iv*) **Fisher's Ideal Formula.** In this formula, the geometric mean of two indices (i.e., Laspeyre's Index and paasche's Index) is taken :

$$p_{01} = \sqrt{\frac{\Sigma P_1 q_0}{\Sigma P_0 q_0} \times \frac{\Sigma P_1 q_1}{\Sigma P_0 q_1}} \times 100 \quad \text{or} \quad P_{01} = \sqrt{L \times P} \times 100$$

where L is Lespeyre's Index and P is paasche's Index.

Example

Comm-odity	Base Year		Current Year		P_0q_0	P_1q_0	P_0q_1	P_1q_1
	P_0	q_0	P_1	q_1				
A	10	5	20	2	50	100	20	40
B	15	4	25	8	60	100	120	200
C	40	2	60	6	80	120	240	360
D	25	3	40	4	75	120	100	160
Total					265 ΣP_0q_0	440 ΣP_1q_0	480 ΣP_0q_1	760 ΣP_1q_1

(*i*) Laspeyre's Formula :

$$p_{01} = \frac{\Sigma P_1 q_0}{\Sigma P_0 q_0} \times 100$$

$$p_{01} = \frac{440}{265} \times 100 = 166.04$$

(ii) Paasche' Formula :

$$p_{01} = \frac{\Sigma P_1 q_1}{\Sigma P_0 q_1} \times 100$$

$$p_{01} = \frac{700}{480} \times 100 = 158.3$$

(iii) Dorbish and Bowley's Formula :

$$p_{01} = \frac{\frac{\Sigma P_1 q_0}{\Sigma P_0 q_0} + \frac{\Sigma P_1 q_1}{\Sigma P_0 q_1}}{2} \times 100 = 162.2$$

$$p_{01} = \frac{\frac{440}{265} + \frac{760}{480}}{2} \times 100 = 162$$

(iv) Fisher's Ideal Formula :

$$p_{01} = \sqrt{\frac{\Sigma P_1 q_0}{\Sigma P_0 q_0} \times \frac{\Sigma P_1 q_1}{\Sigma P_0 q_1}} \times 100$$

$$p_{01} = \sqrt{\frac{440}{265} \times \frac{760}{480}} \times 100 = 162.1$$

4. Weighted Average of Relatives Method:

In this method also different weights are used for the items according to their relative importance.

Formula:

$$P_{01} = \frac{\Sigma RW}{\Sigma W}$$

where ΣW stands for the sum of weights of different commodities :
and ΣR stands for the sum of price relatives.

Commodity	Weights W	Base Prices Year P_0	Current Year Prices P_1	Price Relatives $R = \frac{P_1}{P_0} \times 100$	RW
A	5	10	20	20/10 × 100 = 200.0	1000.0
B	4	15	25	25/15 × 100 = 166.7	666.8
C	2	40	60	60/40 ×100 = 150.0	300.0
D	3	25	40	40/25 × 100 = 160.0	480.0
Total	ΣW=14				ΣRW = 2446.8

$$\text{Index Number } (P_{01}) = \frac{\Sigma RW}{\Sigma W}$$

$$p_{01} = \frac{2446.8}{14} = 174.8$$

Industrial Production Index(IPI)

The index of industrial production measures the change of industrial production relative to a base year.

Unlike WPI and CPI, this index number focuses on production rather than price. It is a useful tool to measure the industrial growth of an economy.

Constructing an IPI

A difference between the construction of an IPI and other indices is the approach towards how weights are considered. For an IPI, weights are assigned according to the output of industries and their contribution to the national income. The following formula is used:

Index number of industrial production = $(\sum RW \div \sum W) \times 100$

Here, R= Ratio of level of production in current year to the level of production in base year

W= Assigned weights

In india The Eight Core industry includes Natural Gas, Coal, Refinery Products, Crude Oil, Cement, Electricity, Steel, and Fertilizers. These industries are termed as the core industries owing to their strong impact on the general economic activities and the other industrial activities.

These industries comprise a total of 40.27% of the total weight of the overall items that are included in the IIP .

The Index of Eight Core Industries highest weight is currently possessed by the Refinery Products Industry. Earlier the highest weightage was given to the Electricity Industry. In the decreasing order of the weightage of these industries, the list is stated as below:

- Refinery Products Industry
- Electricity Industry
- Steel Industry
- Coal Industry
- Crude Oil Industry
- Natural Gas Industry
- Cement Industry
- Fertilizers Industry

Current Figures of Weightage of Eight Core Industries

The weightage figures are stated below in decreasing order.

Industry	Weightage Percentage
Refinery Products Industry	28.04%
Electricity Industry	19.85%
Steel Industry	17.92%
Coal Industry	10.33%
Crude Oil Industry	8.98%
Natural Gas Industry	6.88%
Cement Industry	5.37%
Fertilizers Industry	2.63%

SEQUENCE & SERIES

1. Meaning of a Sequence:-

A sequence is an ordered list of numbers which follow a particular rule or pattern.
The numbers are called terms of the sequence.

Example:
2, 4, 6, 8, 10, … is a sequence of even numbers.

If a sequence is denoted as a1 ,a2 ,a3 ,..., then an is called the **nth term**.

Mathematically:
A sequence is a function from the set of natural numbers to the set of real numbers:

f:N→R,f(n)=an
क्रम **(Sequence)** संख्याओं की एक निश्चित क्रम में सूची होती है जो किसी नियम या पैटर्न का पालन करती है।
प्रत्येक संख्या को **पद** **(Term)** कहा जाता है।

उदाहरण: 2, 4, 6, 8, 10, … यह सम संख्याओं का क्रम है। यदि पदों को a1 ,a2 ,a3 ,... से दर्शाया जाए, तो an को **nवाँ पद** कहा जाता है।

2. Arithmetic Progression (A.P.)

English:
A sequence is called an **Arithmetic Progression** (A.P.) if the difference between any two consecutive terms is constant.
That constant difference is called the **common difference (d)**. a,a+d,a+2d,a+3d,…

Formulas:

- **Sum of n terms (S_n):** Sn =n/2 [2a+(n−1)d]

Example:
Find the sum of first 10 natural numbers.
→ a=1,d=1,n=10,

S(10) =10/2 [2(1)+(10−1)(1)]=5(11)=55

- **nth term (general term)**: $a_n = a\,r^{\,n-1}$.

  (where a = first term, r = common ratio)

3.

- **Sum of first n terms** ($r \neq 1$):

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = a\,\frac{1-r^n}{1-r}.$$

- **Sum to infinity** (only if $|r| < 1$):

$$S_\infty = \frac{a}{1-r}.$$

- **Find common ratio** (if a_1 and a_n given):

$$r = \left(\frac{a_n}{a_1}\right)^{\frac{1}{n-1}}.$$

Geometric Progression (G.P.)

A sequence is called a Geometric Progression (G.P.) if the ratio between any two consecutive terms is constant. That constant ratio is called the common ratio (r).
a, ar, ar2, ar3,…

Formula:-

Given a geometric progression with first term $a_1 = 5$ and common ratio $r = 2$.

(a) Find the 6th term a_6.

(b) Find the sum of the first 6 terms S_6.

Solution

Formulae:

$a_n = a_1 r^{\,n-1}$.

$S_n = a_1 \frac{1-r^n}{1-r}$ (for $r \neq 1$).

(a) $a_6 = 5 \cdot 2^{6-1} = 5 \cdot 2^5$.

Compute $2^5 = 32$. So $a_6 = 5 \times 32 = 160$.

(b) $S_6 = 5 \cdot \frac{1-2^6}{1-2}$.

Compute $2^6 = 64$. So numerator $1 - 64 = -63$. Denominator $1 - 2 = -1$.

Thus $S_6 = 5 \cdot \frac{-63}{-1} = 5 \cdot 63 = 315$.

Answer: $a_6 = 160,\ S_6 = 315$.

Example-1

Example -2

For a G.P. let S_2 (sum of first 2 terms) be 10 and S_4 (sum of first 4 terms) be 40. Find the common ratio r (assume $r > 0$).

Solution

Key identity: $\dfrac{S_{2n}}{S_n} = 1 + r^n$ (derived from $S_n = a\frac{1-r^n}{1-r}$).

Here take $n = 2$. So $\dfrac{S_4}{S_2} = 1 + r^2$.

Compute ratio: $\dfrac{40}{10} = 4$. So $4 = 1 + r^2 \Rightarrow r^2 = 3$.

Hence $r = \sqrt{3} \approx 1.732051$ (positive root as given).

Answer: $r = \sqrt{3} \approx 1.732051$.

Example-3

A firm will receive a royalty of ₹100 at the end of year 1, and each subsequent year the royalty grows by 5% (i.e., growth rate $g = 0.05$). If the discount rate is 12% per year, what is the present value (PV) at time 0 of these perpetual growing royalties?

Solution

This is a growing perpetuity with first payment $A_1 = 100$, growth $g = 0.05$, discount $i = 0.12$, and $i > g$. Formula:

$$PV = \frac{A_1}{i - g}.$$

So $PV = \frac{100}{0.12 - 0.05} = \frac{100}{0.07}.$

Compute: $100/0.07 = 1428.5714285714287\ldots$ Round to two decimals: ₹1428.57.

Answer: $PV \approx$ ₹$1{,}428.57$.

(Economic note: formula assumes payments start at end of first period and growth is constant and perpetual; standard in valuation problems.)

MAXIMA AND MINIMA.

1. Concept Overview / परिचय

In optimization problems, we often want to find the **maximum** or **minimum** value of a function. For example, a consumer wants to **maximize utility**, or a firm wants to **maximize profit** or **minimize cost**.

Optimization problems में हम किसी function का **maximum** या **minimum** value निकालना चाहते हैं। उदाहरण के लिए, उपभोक्ता अपनी **संतुष्टि (utility)** को अधिकतम करना चाहता है या फर्म **लाभ (profit)** को अधिकतम / **लागत (cost)** को न्यूनतम करना चाहती है।

2. Types of Optimization

Type	Description	Example
Unconstrained Optimization	No external restriction or constraint.	Profit maximization without capacity limit
Constrained Optimization	Function is optimized subject to some constraint(s).	Utility maximization subject to budget constraint

Unconstrained Optimization (Single Variable)

Let $f(x)$ be a differentiable function.

- **First Order Condition (FOC):**
  $$\frac{df(x)}{dx} = 0$$
- **Second Order Condition (SOC):**
  $$\frac{d^2f(x)}{dx^2}$$
  - If **< 0**, then **maximum**
  - If **> 0**, then **minimum**
  - If **= 0**, test fails (check higher-order derivatives)

Example:-

Let $f(x) = -x^2 + 6x + 5$

Step 1:

$f'(x) = -2x + 6 = 0 \Rightarrow x = 3$

Step 2:

$f''(x) = -2 < 0 \Rightarrow$ **Maximum at x = 3**

Value: $f(3) = -9 + 18 + 5 = 14$

✅ So, **maximum value = 14 at x = 3**

Unconstrained Optimization (Two Variables)

Let $f(x, y)$ be a function of two variables.

First Order Conditions (FOC):

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

Second Order Conditions (SOC):

Compute the **Hessian determinant (H)**:

$$H = f_{xx} f_{yy} - (f_{xy})^2$$

Condition	Nature of Point
$H > 0$ and $f_{xx} < 0$	**Maximum**
$H > 0$ and $f_{xx} > 0$	**Minimum**
$H < 0$	**Saddle point**
$H = 0$	**Inconclusive**

🧠 **Example 2 (Two Variable Unconstrained):**

Let $f(x, y) = 10x + 5y - x^2 - y^2$

Step 1 (FOC):

$$f_x = 10 - 2x = 0 \Rightarrow x = 5$$

$$f_y = 5 - 2y = 0 \Rightarrow y = 2.5$$

Step 2 (SOC):

$$f_{xx} = -2,\ f_{yy} = -2,\ f_{xy} = 0$$

$$H = (-2)(-2) - 0 = 4 > 0 \text{ and } f_{xx} < 0$$

✅ **Maximum at (5, 2.5)**

Value:

$f(5, 2.5) = 10(5) + 5(2.5) - 25 - 6.25 = 50 + 12.5 - 31.25 = 31.25$

Constrained Optimization (with Lagrange Method)

We want to maximize or minimize $f(x, y)$ **subject to** a constraint $g(x, y) = 0$.

We form the **Lagrangian Function:**

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda[k - g(x, y)]$$

Then, we solve:

$$\frac{\partial \mathcal{L}}{\partial x} = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = 0, \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

Example 3 (Constrained Optimization):

Maximize: U=XY, **Subject to: 2X+Y=100**

Step 1:

$$\mathcal{L} = XY + \lambda(100 - 2X - Y)$$

Step 2 (FOC):

$$\frac{\partial \mathcal{L}}{\partial X} = Y - 2\lambda = 0 \Rightarrow Y = 2\lambda$$

$$\frac{\partial \mathcal{L}}{\partial Y} = X - \lambda = 0 \Rightarrow X = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 100 - 2X - Y = 0$$

Substitute $X = \lambda, Y = 2\lambda$:

$$100 - 2\lambda - 2\lambda = 100 - 4\lambda = 0 \Rightarrow \lambda = 25$$

$$X = 25, Y = 50$$

✅ **Maximum Utility at (X, Y) = (25, 50)**

U = 25 × 50 = 1250

Meaning of a Matrix / मैट्रिक्स का अर्थ

English:

A **matrix** is a rectangular array of numbers (or elements) arranged in rows and columns.
It is usually denoted by a **capital letter (A, B, C, …)** and its elements by **small letters with subscripts**, e.g.,aij .

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Here,

- a_{ij}: element in the *i-th row* and *j-th column*.
- The above matrix has **2 rows** and **3 columns**, so it's a **2×3 matrix**.

TYPES OF MATRICES

🔷 1. Row Matrix (पंक्ति मैट्रिक्स)

- **Definition:**
  A matrix having **only one row** and any number of columns.

$$A = [a_{11}, a_{12}, a_{13}, \ldots, a_{1n}]$$

- **Example:**

$$A = [2, 5, 7]$$

- **Order:** 1 × n
- **In Economics:**
  Used to represent a **set of coefficients or data** for a single observation (e.g., prices of different goods at a single time).

🔷 2. Column Matrix (स्तंभ मैट्रिक्स)

- **Definition:**

  A matrix having **only one column** and any number of rows.

$$B = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{bmatrix}$$

- **Example:**

$$B = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

- **Order:** n × 1
- **In Economics:**

  Used to represent **a variable vector**, like a vector of **quantities demanded, prices, or output levels**.

🔷 3. Square Matrix (वर्गाकार मैट्रिक्स)

- **Definition:**

  A matrix having **equal number of rows and columns**.

$$n \times n$$

- **Example:**

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

- **In Economics:**

  Appears in **input-output models (Leontief model)** and **determinant-based system of equations**.

🔷 4. Rectangular Matrix (आयताकार मैट्रिक्स)

- **Definition:**

  A matrix with **unequal rows and columns**.

$$m \times n, \quad m \neq n$$

- **Example:**

$$A = \begin{bmatrix} 2 & 5 & 7 \\ 3 & 4 & 1 \end{bmatrix}$$

- **In Economics:**

  Used in **data representation**, e.g., demand data for multiple goods and consumers.

🔷 5. Diagonal Matrix (मुख्य विकर्ण मैट्रिक्स)

- **Definition:**

  A **square matrix** in which **all non-diagonal elements are zero**.

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

- **In Economics:**

  Represents **independent relationships**, e.g., cost functions where variables don't interact.

🔷 6. Scalar Matrix (स्केलर मैट्रिक्स)

- **Definition:**

  A **diagonal matrix** where **all diagonal elements are equal**.

$$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

- **Example:**

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

- **In Economics:**

  Represents **uniform scaling** — e.g., when all sectors grow at the same rate.

🔷 7. Identity Matrix / Unit Matrix (एकक मैट्रिक्स)

- **Definition:**

  A **square matrix** in which **all diagonal elements are 1**, and **others are 0**.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Property:**

  $A \times I = A$ and $I \times A = A$

- **In Economics:**

  Used in **Leontief inverse $(I – A)^{-1}$** in Input-Output analysis.

🔷 8. Zero or Null Matrix (शून्य मैट्रिक्स)

- **Definition:**

  A matrix in which **all elements are zero**.

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- **Property:**

  $A + O = A$

- **In Economics:**

  Represents **no effect** or **zero interactions** among variables.

🔷 9. Triangular Matrix (त्रिकोणीय मैट्रिक्स)

(a) Upper Triangular Matrix

All elements **below** the main diagonal are zero.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 7 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) Lower Triangular Matrix

All elements **above** the main diagonal are zero.

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 6 & 7 \end{bmatrix}$$

- **In Economics:**

  Used in **decomposition of systems** (LU decomposition) and solving **simultaneous equations** efficiently.

🔷 10. Symmetric Matrix (साम्य मैट्रिक्स)

- **Definition:**
  A **square matrix** that satisfies $A' = A$ (transpose equals itself).

$$a_{ij} = a_{ji}$$

- **Example:**

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

- **In Economics:**
  Appears in **variance-covariance matrices**, **quadratic forms**, and **optimization** problems.

🔷 11. Skew-Symmetric Matrix (विकृत-साम्य मैट्रिक्स)

- **Definition:**
  A **square matrix** that satisfies $A' = -A$.

$$a_{ij} = -a_{ji}$$

- **Example:**

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

- **Property:**
  Diagonal elements are always **zero**.
- **In Economics:**
  Used in **mathematical modeling** and **differential systems**.

🔷 12. Singular and Non-Singular Matrices

- **Singular Matrix:**
  Determinant = 0 → Not invertible.
- **Non-Singular Matrix:**
  Determinant ≠ 0 → Invertible.
- **In Economics:**
  When solving equations like $AX = B$,
  - If **A** is non-singular → unique solution.
  - If **A** is singular → no or infinite solutions.

🔷 13. Idempotent Matrix

- **Definition:**
  A matrix A such that $A^2 = A$.
- **Example:**

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- **In Economics:**
  Used in **econometrics** — projection matrix in **OLS estimation** $P = X(X'X)^{-1}X'$ is idempotent.

🔷 14. Involutory Matrix

- **Definition:**
  $A^2 = I$
- **Example:**

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- **In Economics:**
  Rare, but appears in **transformation models** or **reversible mappings**.

🔷 15. Orthogonal Matrix

- **Definition:**
  A matrix A such that $A'A = AA' = I$.
  (Transpose = Inverse)
- **In Economics:**
  Used in **Principal Component Analysis (PCA)** and **factor analysis** for orthogonal transformations.

Basic Operations on Matrices

1. Addition and Subtraction of Matrices

Rule: Two matrices can be added or subtracted only if they are of the same order (i.e., same number of rows and columns).

$$A = [a_{ij}], \quad B = [b_{ij}]$$

$$A + B = [a_{ij} + b_{ij}]$$

$$A - B = [a_{ij} - b_{ij}]$$

Example:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 7 & 4 \\ 3 & 4 \end{bmatrix}$$

Economic Application:

- Combining **cost matrices**, **input-output data**, or **utility values** from different sectors.
  Example: Total production of two firms across two goods.

2. Scalar Multiplication

Definition: Each element of a matrix is multiplied by a scalar constant k.

$$kA = [k \times a_{ij}]$$

Example:

$$2 \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

Economic Application:

- Representing **changes in price or output levels** due to inflation or scaling factors.
  Example: If inflation rate doubles the cost, the cost matrix is multiplied by 2.

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3. Matrix Multiplication

Rule: If A is of order $m \times n$ and B is $n \times p$, then AB is $m \times p$.

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Example:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} (2 \times 4 + 1 \times 2) & (2 \times 5 + 1 \times 1) \\ (0 \times 4 + 3 \times 2) & (0 \times 5 + 3 \times 1) \end{bmatrix} = \begin{bmatrix} 10 & 11 \\ 6 & 3 \end{bmatrix}$$

Economic Applications:

- **Input-Output Analysis (Leontief Model):**
  The output vector = $(I - A)^{-1} D$, where
  A = input coefficient matrix,
  D = final demand vector,
  $(I - A)^{-1}$ = Leontief inverse.
- **National Income Accounting:** $Y = AX$
- **Production and Consumption relations**

4. Transpose of a Matrix

Definition: The transpose of matrix A (denoted A' or A^T) is obtained by interchanging rows and columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Economic Application:

- Used to switch between **row and column representations** of data.
  Example: Changing data from "goods by industries" to "industries by goods."

5. Determinant of a Matrix

Definition: A scalar value that can be computed from a square matrix; denoted as $|A|$.

For a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad |A| = ad - bc$$

Economic Application:

- Used in **solving simultaneous linear equations** in equilibrium analysis.
  Example: Market equilibrium of two goods using **Cramer's rule**.

6. Inverse of a Matrix

Definition: For a square matrix A, if there exists a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

then A^{-1} is the **inverse of** A.

Formula for 2×2 matrix:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Economic Applications:

- **Leontief Input-Output Model:**
  $X = (I - A)^{-1}D$ → used to find total output required to meet final demand.
- **Solving Economic Equilibrium Equations** (like IS-LM or General Equilibrium systems).

Input-Output Analysis

Input-Output Analysis is an analytical framework developed by **Wassily Leontief** to study the interdependence among different sectors of an economy.
It shows how the output of one industry is used as an input by another and helps to analyze the flow of goods and services within the economy.

It is represented through an **Input-Output Table (or Matrix)** which records the transactions between industries and helps in understanding how changes in demand in one sector affect output levels in others.

Key Idea:
Each industry both produces output and uses inputs from other industries — creating an interlinked system of production.

Uses:

- To study **inter-industry relationships**
- To estimate **impact of policy changes or investments**
- To analyze **structural changes** in the economy

इनपुट-आउटपुट विश्लेषण (Input-Output Analysis) एक विश्लेषणात्मक पद्धति है जिसे **वासिली लेओनटीफ़ (Wassily Leontief)** ने विकसित किया था।
यह अर्थव्यवस्था के विभिन्न क्षेत्रों (industries/sectors) के **आपसी निर्भरता (interdependence)** को समझने के लिए प्रयोग किया जाता है।

यह दर्शाता है कि एक उद्योग का उत्पादन दूसरे उद्योग के लिए इनपुट के रूप में कैसे उपयोग होता है।
इसे **इनपुट-आउटपुट तालिका (Input-Output Table या Matrix)** के रूप में प्रस्तुत किया जाता है, जिसमें उद्योगों के बीच लेन-देन को दर्शाया जाता है।

मुख्य विचार:
हर उद्योग अन्य उद्योगों से इनपुट लेता है और स्वयं आउटपुट प्रदान करता है — इस प्रकार अर्थव्यवस्था में एक पारस्परिक संबंध (interconnected system) बनता है।

उपयोग:

- **उद्योगों के बीच संबंधों** का अध्ययन करने के लिए
- **नीति परिवर्तन या निवेश** के प्रभाव का आकलन करने के लिए
- अर्थव्यवस्था की **संरचनात्मक परिवर्तन** को समझने के लिए

Assumptions of Input-Output Analysis

(A) In English

1. **Linear Production Function:**
   The relationship between inputs and outputs is linear. This means output increases in fixed proportion with inputs — there are constant returns to scale.

2. **Constant Technology:**
   The technology used in production does not change during the analysis period.

3. **Fixed Input Coefficients:**
   Each industry uses inputs from other industries in fixed proportions — no substitution among inputs is possible.

4. **Two-Sector or Multi-Sector Economy:**
   The economy is divided into sectors (industries), each producing one homogeneous commodity.

5. **No Externalities:**
   Production and consumption activities of one industry do not affect others except through market transactions.

6. **Supply Equals Demand (Equilibrium Condition):**
   Total output of each industry equals the total demand for that industry's product (both intermediate and final demand).

7. **Closed and Open Models:**
   - In a **closed model**, households are included as both producers (of labor) and consumers.
   - In an **open model**, households are treated as external to the system.

8. **No Inventory Changes:**
   It assumes that all produced goods are sold — there are no unsold stocks or changes in inventory.

9. **Perfect Competition:**
   Prices are assumed to be constant, determined by the equality of supply and demand.

(Input-Output विश्लेषण की मान्यताएँ)

1. **रेखीय उत्पादन फलन (Linear Production Function):**
   इनपुट और आउटपुट के बीच संबंध रेखीय होता है — अर्थात् इनपुट बढ़ने पर आउटपुट समान अनुपात में बढ़ता है (Constant Returns to Scale)।

2. **प्रौद्योगिकी स्थिर है (Constant Technology):**
   अध्ययन की अवधि में उत्पादन तकनीक में कोई परिवर्तन नहीं होता।

3. **नियत इनपुट गुणांक (Fixed Input Coefficients):**
प्रत्येक उद्योग अन्य उद्योगों से इनपुट को निश्चित अनुपात में उपयोग करता है — इनपुट्स के बीच प्रतिस्थापन संभव नहीं है।

4. **दो-क्षेत्रीय या बहु-क्षेत्रीय अर्थव्यवस्था (Two-Sector or Multi-Sector Economy):**
अर्थव्यवस्था को विभिन्न क्षेत्रों (Industries) में बाँटा जाता है, और प्रत्येक उद्योग एक समान उत्पाद बनाता है।

5. **कोई बाह्य प्रभाव नहीं (No Externalities):**
किसी उद्योग का उत्पादन या उपभोग अन्य उद्योगों को प्रभावित नहीं करता, सिवाय बाज़ार लेनदेन के माध्यम से।

6. **आपूर्ति = मांग (संतुलन स्थिति):**
प्रत्येक उद्योग का कुल उत्पादन उसकी कुल मांग (मध्यवर्ती + अंतिम मांग) के बराबर होता है।

7. **बंद और खुला मॉडल (Closed and Open Models):**
   - **बंद मॉडल (Closed Model):** गृहस्थियाँ (Households) उत्पादक और उपभोक्ता दोनों होती हैं।
   - **खुला मॉडल (Open Model):** गृहस्थियाँ सिस्टम के बाहर मानी जाती हैं।

8. **कोई भंडार परिवर्तन नहीं (No Inventory Changes):**
यह माना जाता है कि सभी उत्पाद बिक जाते हैं — कोई अधिशेष या स्टॉक नहीं बचता।

9. **संपूर्ण प्रतिस्पर्धा (Perfect Competition):**
कीमतें स्थिर मानी जाती हैं और आपूर्ति व मांग की समानता से निर्धारित होती हैं।

Basic Equations of the Model:

The total output of each sector = Intermediate demand + Final demand

$$X = AX + D$$

Rearranging,

$$X - AX = D$$

$$(I - A)X = D$$

$$X = (I - A)^{-1}D$$

Here,

- X: Column vector of total outputs
- A: Matrix of input coefficients $[a_{ij}]$
- D: Column vector of final demand
- $(I - A)^{-1}$: **Leontief Inverse Matrix**

4. Example

Let the economy have **two sectors**:

1️⃣ Agriculture (1)

2️⃣ Manufacturing (2)

Given:

$$A = \begin{bmatrix} 0.3 & 0.2 \\ 0.4 & 0.1 \end{bmatrix}, \quad D = \begin{bmatrix} 100 \\ 80 \end{bmatrix}$$

Find **total output (X)**.

Step 1: Write formula

$$X = (I - A)^{-1} D$$

Step 2: Compute (I - A)

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.2 \\ 0.4 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.2 \\ -0.4 & 0.9 \end{bmatrix}$$

Step 3: Find inverse of (I - A)

$$(I - A)^{-1} = \frac{1}{(0.7)(0.9) - (-0.2)(-0.4)} \begin{bmatrix} 0.9 & 0.2 \\ 0.4 & 0.7 \end{bmatrix}$$

$$= \frac{1}{0.63 - 0.08} \begin{bmatrix} 0.9 & 0.2 \\ 0.4 & 0.7 \end{bmatrix} = \frac{1}{0.55} \begin{bmatrix} 0.9 & 0.2 \\ 0.4 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 1.636 & 0.364 \\ 0.727 & 1.273 \end{bmatrix}$$

Step 4: Multiply by D

$$X = \begin{bmatrix} 1.636 & 0.364 \\ 0.727 & 1.273 \end{bmatrix} \begin{bmatrix} 100 \\ 80 \end{bmatrix}$$

$$X = \begin{bmatrix} (1.636 \times 100) + (0.364 \times 80) \\ (0.727 \times 100) + (1.273 \times 80) \end{bmatrix} = \begin{bmatrix} 165.7 \\ 229.8 \end{bmatrix}$$

✅ **Total Outputs:**

- Agriculture = **165.7**
- Manufacturing = **229.8**

Interpretation (अर्थ)

An increase in final demand of **100 (Agri)** and **80 (Manuf.)** requires total outputs of **165.7** and **229.8**, respectively, to satisfy both **intermediate** and **final demands**.

Difference & differential Equation

Difference Equations (Discrete Time Models)

A **Difference Equation** is an equation that expresses the relationship between the value of a variable at one point in time and its value at another point in time (usually at the next or previous period).

It is used to describe how an **economic variable changes over discrete time intervals** (like years, months, or quarters).

General form: yt =f(yt−1 ,yt−2 ,…)

Here,

yt = value of the variable (like income, output, or capital) at time t

yt−1 = value at the previous time period

f(⋅) = functional relationship describing how the variable evolves over time

Example 1: Simple First-Order Linear Difference Equation:

$$Y_t = 0.8Y_{t-1} + 100$$

Here,

- Y_t: National income in year t
- Y_{t-1}: National income in the previous year
- 0.8: Marginal propensity to consume (MPC)
- 100: Autonomous expenditure

Step 1️⃣: Assume initial income

$$Y_0 = 500$$

Step 2️⃣: Compute next few periods

$$Y_1 = 0.8(500) + 100 = 50($$
$$Y_2 = 0.8(500) + 100 = 50($$

This system reaches **equilibrium** immediately at Y_t ↓ 500.

Example 2: Non-equilibrium case

If $Y_0 = 200$,

$$Y_1 = 0.8(200) + 100 = 260$$
$$Y_2 = 0.8(260) + 100 = 308$$
$$Y_3 = 0.8(308) + 100 = 346.4$$

As $t \to \infty$,

$$Y_t \to 500$$

So the **steady-state (equilibrium) value** is

$$Y^* = \frac{100}{1 - 0.8} = 500$$

📈 Economic Interpretation:

This difference equation shows how **income adjusts over time** in a simple Keynesian model where consumption depends on past income. The system converges to a steady-state equilibrium level of income.

A **second-order difference equation** is a discrete-time dynamic model in which the **current value of a variable** depends on its **two previous values**.

It takes the general form:

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + b$$

Where:

- y_t = current value of the variable (e.g., income, output, consumption, etc.)
- y_{t-1}, y_{t-2} = first and second lagged values (previous periods)
- a_1, a_2 = constants (coefficients)
- b = constant term or external shock

💡 Interpretation:

This equation explains how the present value y_t is determined by its **two past values** — capturing **momentum**, **oscillation**, or **stability** over time in economic models.

Suppose the difference equation is:

$$y_t = 0.6y_{t-1} - 0.08y_{t-2}$$

with initial conditions:

$$y_0 = 100, \quad y_1 = 90$$

Let's find y_2 and y_3.

Step 1: For $t = 2$:

$$y_2 = 0.6(90) - 0.08(100) = 54 - 8 = 46$$

Step 2: For $t = 3$:

$$y_3 = 0.6(46) - 0.08(90) = 27.6 - 7.2 = 20.4$$

✅ So,

$$y_0 = 100,\ y_1 = 90,\ y_2 = 46,\ y_3 = 20.4$$

This shows how the system evolves **discretely** over time using past values.